Integration: Reduction Formulas

Any positive integer power of $\sin x$ can be integrated by using a reduction formula.

Example

Prove that for any integer $n \geq 2$,

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

Solution. We use integration by parts.

- Let $u = \sin^{n-1} x$ and $dv = \sin x \, dx$. Then, $du = (n-1) \sin^{n-2} x \cos x \, dx$ and we can use $v = -\cos x$.
- So,

$$\begin{aligned}
\int \sin^n x \, dx &= \int \sin^{n-1} x \sin x \, dx \\
&= \sin^{n-1} x(-\cos x) - \int (-\cos x)(n-1) \sin^{n-2} x \cos x \, dx \\
&= \text{-}\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx \\
&= \text{-}\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \,(1 - \sin^2 x) \, dx \\
&= \text{-}\sin^{n-1} x \cos x + (n-1) \int \left(\sin^{n-2} x - \sin^n x\right) \, dx \\
&= \text{-}\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx
\end{aligned}$$

- Re-arranging, we get $n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$.
- Dividing both sides by n, we get $\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$.

Example

Use the reduction formula to find the integrals of $\sin^2 x, \sin^3 x, \sin^4 x$.

Solution. We recall the reduction formula proved above. For $n \geq 2$,

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

- With $n = 2$,

$$\begin{aligned} \int \sin^2 x \, dx &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int 1 \, dx \\ &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} x + C. \end{aligned}$$

- With $n = 3$,

$$\begin{aligned} \int \sin^3 x \, dx &= -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx \\ &= -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C. \end{aligned}$$

- With $n = 4$,

$$\begin{aligned} \int \sin^4 x \, dx &= -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx \\ &= -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \left(-\frac{1}{2} \sin x \cos x + \frac{1}{2} x \right) + C. \\ &= -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C. \end{aligned}$$

Example

Express $\sin^3 x\,\cos^4 x$ as a sum of constant multiples of $\sin x$. Hence, or otherwise, find the integral of $\sin^3 x\,\cos^4 x$.

Solution. Since $\cos^2 x = 1 - \sin^2 x$,

- $\cos^6 x = (\cos^2 x)^4 = (1 - \sin^2 x)^2 = 1 - 2\sin^2 x + \sin^4 x$.
- So, $\sin^3 x \cos^6 x = \sin^3 \left(1 - 2\sin^2 x + \sin^4 x\right) = \sin^3 x - 2\sin^5 x + \sin^7 x$.

For the sake of simplicity, we will denote $\int \sin^n x\, dx$ by I_n.

- The reduction formula reads $I_n = -\frac{1}{n}\sin^{n-1} x\,\cos x + \frac{n-1}{n} I_{n-2}$.
- By using the reduction formula, we get

$$\begin{aligned}
\mathrm{I}_1 &= \text{-}\cos x + C\\
\mathrm{I}_3 &= \text{-}\frac{1}{3}\sin^2 x\,\cos x - \frac{2}{3}\cos x + C\\
\mathrm{I}_5 &= \text{-}\frac{1}{5}\sin^4 x\,\cos x - \frac{4}{15}\sin^2 x\,\cos x - \frac{8}{15}\cos x + C\\
\mathrm{I}_7 &= \text{-}\frac{1}{7}\sin^6 x\,\cos x - \frac{6}{35}\sin^4 x\,\cos x - \frac{8}{35}\sin^2 x\,\cos x - \frac{16}{35}\cos x + C
\end{aligned}$$

- Integrating both sides of the identity

$$\sin^3 x \cos^4 x = \sin^3 x - 2\sin^5 x + \sin^7 x,$$

we get

$$\int \sin^3 x \cos^4 x\, dx = I_3 - 2I_5 + I_7 = -\frac{1}{7}\sin^6 x \cos x + \frac{8}{35}\sin^4 x \cos x - \frac{1}{35}\sin^2 x \cos x - \frac{2}{35}\cos x + C.$$

Remarks.

- One can use integration by parts to derive a reduction formula for integrals of powers of cosine:
$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \, \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$
One can integrate all positive integer powers of $\cos x$.
- By using the identity $\sin^2 = 1 - \cos^2 x$, one can express $\sin^m x \cos^n x$ as a sum of constant multiples of powers of $\cos x$ if m is even.
- In view of this and our previous examples, we can integrate $\sin^m x \cos^n x$ as long as m and/or n is even.
- If both m and n are odd, then we need a different approach. It turns out that either of the substitutions $u = \sin x$ and $u = \cos x$ will work.

Example

- If we use $u = \sin x$, then $du = \cos x \, dx$, and since $\cos^4 x = (1 - \sin^2 x)^2 = (1 - u^2)^2$,
$$\int \sin^3 x \cos^5 x dx = \int \sin^3 \cos^4 x \, dx = \int u^3 (1 - u^2)^2 du = \int (u^3 - 2u^5 + u^7) du = \dots$$
- If we use $u = \cos x$, then $du = -\sin x \, dx$, and since $\cos^2 x = 1 - \sin^2 x = 1 - u^2$,
$$\int \sin^3 x \cos^5 x dx = \int (-\sin^2 x) \cos^5 x (-\sin x) dx = \int -(1 - u^2) u^5 du = \int (-u^5 + u^7) du = \dots$$

- Convince yourself this: If m is odd, then $u = \cos x$ will work (even if n is even). If n is odd, then $u = \sin x$ will work (even if m is even).
- All functions of the form $\sin^m x \cos^n x$ can be integrated.

Let's consider integrals of the form $\int \sec^m x \tan^n x \, dx$.

Example

Prove that for any integer $n \geq 2$,

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx.$$

Use this and the fact that $\int \tan x \, dx = \ln|\sec x| + C$ to find the integrals of $\tan^2 x, \tan^3 x, \tan^4 x$ and $\tan^5 x$.

- Using the identity $\tan^2 x = \sec^2 x - 1$,
$$\int \tan^n x \, dx = \int \tan^{n-2} x tan^2 x \, dx = \int \tan^{n-2} x (sec^2 x - 1) \, dx = \int \tan^{n-2} x sec^2 x \, dx - \int \tan^{n-2} x \, dx.$$
- Using the substitution $u = \tan x$, we have $du = \sec^2 x \, dx$. So,
$$\int \tan^{n-2} x \sec^2 x \, dx = \int u^{n-2} du = \frac{1}{n-1} u^{n-1} + C = \frac{1}{n-1} \tan^{n-1} x + C.$$
The reduction formula is proved.
- With $n = 2$, we have $\int \tan^2 x \, dx = \tan x - \int 1 dx = \tan x - x + C$.
- With $n = 3$, we have $\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \int \tan x \, dx = \frac{1}{2} \tan^2 x - \ln|\sec x| + C$.
- With $n = 4$, we have $\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \int \tan^2 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C$.
- With $n = 5$, we have $\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \int \tan^3 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \ln|\sec x| + C$.

Remarks.

- One can integrate all positive integer powers of $\tan x$.
- One can derive a reduction formula for $\sec x$ by integration by parts.
- Using the reduction formula and the fact $\int \sec x \, dx = \ln|\sec x + \tan x| + C$, we can integrate all positive integer powers of $\sec x$.
- Similar strategies used for $\sin^m x \cos^n x$ can be formulated to integrate all functions of the form $\sec^m x \tan^n x$.

Further Examples and Exercises

- Prove the reduction formula for integrals of powers of $\cos x$:
$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \, \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$
Use it to find the integrals of $\cos^2 x, \cos^3 x, \cos^4 x, \cos^5 x, \cos^6 x$.
- Express $\sin^4 x \, \cos^6 x$ as a sum of constant multiples of $\cos x$. Hence, or otherwise, find the integral of $\sin^4 x \, \cos^6 x$.
- Use integration by parts to find a reduction formula for integrals of positive integer powers of $\sec x$.
- Find the following integrals.
$$\int \sin^5 x \, \cos^2 x \, dx, \quad \int \cos^4 x \, \sin^2 x \, dx, \quad \int \sin^4 x \cos^4 x \, dx$$
- Find the following integrals.
$$\int \tan^3 x \, dx, \quad \int \sec^5 x \, dx, \quad \int \tan^4 x \, dx, \quad \int \sec^3 x \tan^2 x \, dx, \quad \int \sec^4 x \tan^3 x \, dx$$